Constrained optimization with Equality Constraint: [NLPP] (or) Multivariate Optimization with equality Constraints

~~In non-Linear Programming~~

Problem:

Obtain the set of necessary conditions for non-Linear Programming Problem (NLPP)

Minimize $z = kx^{-1}y^{-2}$

Subject to the constraints $x^2+y^2-a^2=0$ with $x \geq 0$, $y \geq 0$ and hence find the minimum value of z.

Solution: The Lagrangian function is

$$L(x,y,\lambda) = kx^{-1}y^{-2} + \lambda(x^2+y^2-a^2)$$

The necessary condition for the minimum of $f(x,y)$ is

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

$$L = kx^{-1}y^{-2} + \lambda(x^2+y^2-a^2)$$

$$\frac{\partial L}{\partial x} = k(-1)x^{-2}y^{-2} + \lambda(2x)$$

$$= -kx^{-2}y^{-2} + 2\lambda x$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow -kx^{-2}y^{-2} + 2\lambda x = 0 \longrightarrow (1)$$

Ily $\frac{\partial L}{\partial y} = kx^{-1}(-2)y^{-3} + \lambda(0+2y-0)$

$$= -2kx^{-1}y^{-3} + 2\lambda y$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow -2kx^{-1}y^{-3} + 2\lambda y = 0 \rightarrow (2)$$

$$\frac{\partial L}{\partial \lambda} = 0 + (1)(x^2+y^2-a^2)$$

$$= x^2+y^2-a^2$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2+y^2-a^2 = 0 \rightarrow (3)$$

From equation (1),

$$2\lambda x = k x^{-2} y^{-2}$$

$$\Rightarrow 2\lambda = \frac{k x^{-2} \cdot y^{-2}}{x^1}$$

$$\boxed{2\lambda = k x^{-3} y^{-2}}$$

From equation (2)

$$2\lambda y = 2k x^{-1} y^{-3}$$

$$2\lambda = \frac{2kx^{-1}y^{-3}}{y^1} = 2kx^{-1}y^{-4}$$

$$\boxed{2\lambda = 2kx^{-1}y^{-4}}$$

$$\Rightarrow k x^{-3} y^{-2} = 2kx^{-1}y^{-4} = 2\lambda$$

$$\Rightarrow \cancel{k} x^{-3} y^{-2} = 2\cancel{k} x^{-1} y^{-4}.$$

$$\Rightarrow y^4 \cdot y^{-2} = 2x^{-1} x^3.$$

$$\Rightarrow y^2 = 2x^2$$

$$\Rightarrow x^2 = y^2/2$$

$$\Rightarrow x = y/\sqrt{2}$$

Sub. $x = y/\sqrt{2}$ in eq (3) $\Rightarrow \left(\frac{y}{\sqrt{2}}\right)^2 + y^2 = a^2$

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$$\frac{y^2}{2} + y^2 = a^2$$

$$\frac{3y^2}{2} = a^2$$

$$y^2 = \frac{2a^2}{3}$$

$$\boxed{y = \sqrt{\frac{2}{3}}\ a}$$

$$\therefore \quad x = y/\sqrt{2}$$

$$x = \frac{\sqrt{\frac{2}{3}}\ a}{\sqrt{2}} = \frac{a}{\sqrt{3}}.$$

$$\boxed{x = \frac{a}{\sqrt{3}}}$$

$\therefore$ Critical point ~~is~~ or stationary point is

$$\left(\frac{a}{\sqrt{3}}, \sqrt{\frac{2}{3}}\ a\right).$$

$$\therefore \text{ Min. } z = k x^{-1} y^{-2}$$

$$= k \left(\frac{a}{\sqrt{3}}\right)^{-1} \left(\sqrt{\frac{2}{3}}\ a\right)^{-2}$$

$$= k \left(\frac{\sqrt{3}}{a}\right) \frac{1}{\left(\sqrt{\frac{2}{3}}\ a\right)^2}$$

$$= k \frac{\sqrt{3}}{a} \cdot \frac{1}{\left(\frac{2}{3}\right) a^2}$$

$$= \frac{k\sqrt{3}\ 3}{a \cdot 2a^2} = \frac{3\sqrt{3}\ k}{2a^3}.$$

$$\frac{y^2}{2} + y^2 = a^2$$

$$\frac{3y^2}{2} = a^2$$

$$y^2 = \frac{2a^2}{3}$$

$$\boxed{y = \sqrt{\frac{2}{3}}\ a}$$

$\therefore$ $$x = y/\sqrt{2}$$

$$x = \frac{\sqrt{\frac{2}{3}}\ a}{\sqrt{2}} = \frac{a}{\sqrt{3}}.$$

$$\boxed{x = \frac{a}{\sqrt{3}}}$$

$\therefore$ Critical point ~~is~~ or stationary point is

$$\left(\frac{a}{\sqrt{3}}, \sqrt{\frac{2}{3}}\ a\right).$$

$\therefore$ Min. $z = k x^{-1} y^{-2}$

$$= k\left(\frac{a}{\sqrt{3}}\right)^{-1}\left(\sqrt{\frac{2}{3}}\ a\right)^{-2}$$

$$= k\left(\frac{\sqrt{3}}{a}\right)\frac{1}{\left(\sqrt{\frac{2}{3}}\ a\right)^2}$$

$$= k\frac{\sqrt{3}}{a}\cdot\frac{1}{\left(\frac{2}{3}\right)a^2}$$

$$= \frac{k\sqrt{3}\ \ 3}{a\cdot 2a^2} = \frac{3\sqrt{3}\,k}{2a^3}.$$

$$\boxed{\text{Min } z = \frac{3\sqrt{3}K}{2a^3}.}$$

Problem 2:
Obtain the set of necessary conditions for the NLPP. Maximize $z = x_1^2 + 3x_2^2 + 5x_3^2$ subject to the constraints. $5x_1 + 2x_2 + x_3 = 5$, $x_1 + x_2 + 3x_2 = 2$,
$x_1, x_2, \geq 0$.

Solution:
Construct Lagrangian function by introducing λ_1 and λ_2 '2' Lagrangian multipliers ~~for~~ for each of the constraints we get.

$$L = x_1^2 + 3x_2^2 + 5x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

The necessary conditions are

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - \lambda_1(1 + 0 + 0 - 0) - \lambda_2(5(1) + 0 + 0 - 0)$$

$$2x_1 - \lambda_1 - 5\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 0 + 3(2x_2) + 0 - \lambda_1(0 + 1 + 0 - 0) - \lambda_2(0 + 2) = 0$$

$$6x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 5(2x_3) - \lambda_1(3) - \lambda_2(1)$$

$$10x_3 - 3\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0, \Rightarrow -(5x_1 + 2x_2 + x_3 - 5) = 0$$

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Minimize $f(x,y) = 2y + x$

Subject to $g(x,y) = y^2 + xy - 1 = 0$

Solve:

$$L(x,y,\lambda) = 2y + x + \lambda(y^2 + xy - 1)$$

$$\frac{\partial L}{\partial x} = 1 + \lambda(y) = 0, \quad \lambda y = -1, \quad \boxed{y = -1/\lambda}$$

$$\frac{\partial L}{\partial y} = 2 + \lambda(2y + x) = 0$$

$$\Rightarrow 2 + 2\lambda y + \lambda x = 0$$

$$\Rightarrow 2 + 2\lambda(-1/\lambda) + \lambda x = 0$$

$$\Rightarrow 2 - 2 + \lambda x = 0$$

$$\Rightarrow \boxed{x = 0}$$

$$\frac{\partial L}{\partial \lambda} = y^2 + xy - 1$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow y^2 xy - 1 = 0$$

sub, $x = 0$,

$$y^2 - 1 = 0$$

$$y = \pm 1$$

when $y = 1$

$y = -1/\lambda$

$\lambda = -y$

$\lambda = -1$

$\therefore$ points are $(x, y, \lambda) = (0, 1, -1)$

When $y = -1$,
we get $\lambda = -1/y$

$= \frac{-1}{-1}$

$\boxed{\lambda = 1}$

$\therefore$ Points $(0, -1, -1)$

$\therefore$ At $(0,1)$ $L = 2y + x + \lambda(x^2 + xy - 1)$

$L = 2(1) + 0 + (-1)[0 + 0 - 1]$

$= 2 + 1$

$\boxed{L = 3}$

At $(0,-1)$ $L = 2(-1) + 0 - 1((-1)^2 0 - 1)$

$= -2 - 1$

$= -3$

$f(x,y) = 2y + x$

$f(0,1) = 2(1) + 0 = 2$

$\boxed{f(0,-1) = 2(-1) = -2}$ (solution)